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Quantum relativistic theory of an electron in terms of local observables

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Abstract. Dirac equation is reformulated in terms of real local observables, which are mean values $(\overline{\Phi}(\mathbf{x})O\Phi(\mathbf{x}))$ of the wave function $\Phi(\mathbf{x})$. The quadrivector current is shown to be a function of the potential vector and of other local observables. The equations describe the evolution of a four dimensional system **T, X, Y, Z**, and of two scalars, in the coordinate system *ct, x, y, z*. The current is proportional to the **T** vector. The **Z** vector is associated with the spin of the electron. Energy and gauge transformations correspond to rotations in the plane (**X, Y**). In the presence of a static field, the (real) solutions of the equations appear as eigenfunctions associated with energy eigenvalues.

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1 Introduction

Physicists have known for years that a representation of the Dirac equation in terms of local observables is possible [1,2] but complicated, and here a much simpler representation is derived. More explicitly, relations involving only quantities of the form $(\overline{\Phi}(\mathbf{x})O\Phi(\mathbf{x}))$, where $\mathbf{x}$ represents the local coordinates and where O is a product of γ^j 's, can be found, and these relations are equivalent to the original Dirac equation. In this way, a *closed* set of new equations has been obtained, a possibility which is not at all obvious and which does not generally occur in statistical mechanics. Unfortunately, these equivalent expressions are rather complicated, since the phase of the wave function and the corresponding gauge transformations, are not observables in a strict sense, and have to be eliminated. However the present article will show that *this complication can be avoided* and that local observables can be introduced and gauge transformations can be retained, simultaneously. Actually, before dealing with quantum mechanics, we show, in the first section, how the same idea applies in classical physics. In this way, the reader will immediately understand our point of view.

What are the motivations of this research? Our far aim is of course to find a correct interpretation of quantum mechanics, but we shall not deal with this matter here. However, while remaining strictly orthodox, this article provides a much clearer view of the physics of relativistic quantum mechanics, in particular by giving an interpretation of spin and of the so called "Schrödinger Zitterbewegung."

2 Classical equations and transformation

In the following, we assume that $\tau = ct$ and that $g^{\tau\tau} = -g^{xx} = -g^{yy} = -g^{zz} = 1$.

The classical motion of an electron in an electromagnetic field is given by:

$$m(dv_k/ds) = mv^j\delta_j v_k = e\ F_{kj}v^j \qquad (1)$$

where v^j is a component of the velocity of the electron (in four dimensions), e is the charge of the electron and ds is the differential of the proper time (an invariant distance in space-time). The electromagnetic field F_{jk} is given in terms of the potential vector A_k by

$$F_{jk} = \partial_j A_k - \partial_k A_j. \qquad (2)$$

On the other hand, we have

$$v^j v_j = c^2. \qquad (3)$$

Equations (1, 3) define the motion of the electron: in a form which we call the *Galileo-Newton* formalism. By combining these equations, we can write equation (1) as follows

$$v^j[m(\partial_j v_k - \partial_k v_j) + e(\partial_j A_k - \partial_k A_j)] = 0. \qquad (4)$$

The preceding equation suggests the following one

$$m(\partial_j v_k - \partial_k v_j) + e(\partial_j A_k - \partial_k A_j) = 0. \qquad (5)$$

This equation entails equation (4) but is not equivalent to it, since it defines a field and a special one (and not only a trajectory). This approach corresponds to the *Hamilton-Jacobi* formalism. The inequivalence of the two formalisms is not generally emphasized but can be verified by reading carefully classical books [3]. Moreover it has been thoroughly discussed by the author elsewhere [4]. Fortunately, the classical limit of quantum equations correspond with the Hamilton-Jacobi formalism and not with the Galileo-Newton one [4]. Therefore, we must take equation (5) as a starting point.

Integrating it, we obtain

$$mv_k = -e(A_k - \partial_k \chi) \tag{6}$$

where χ is an arbitrary function equivalent to equation (5). We remark immediately that this equation deals with physical quantities but is simultaneously invariant by gauge transformations. Equations (6) and (3) can now be considered as the basic equations which define the motion of an electron in an electromagnetic field. Consequently, equation (6) must be generalized in the quantum case and this aim will be reached in the following

3 Kinetic framework associated with an electron in relativistic quantum mechanics

The algebra of the γ^j constitutes the basis of the Dirac theory. Local observables *i.e.* tensors are associated with the elements of this algebra. Thus, before considering dynamics, we must recall the structure of these tensors and their relationships. The symbols γ^j, with $j = \tau,\ x,\ y,\ z$, represent 4×4 matrices. We have

$$\begin{aligned} \gamma^j\gamma^k + \gamma^k\gamma^j &= 2g^{jk} I \\ (\gamma^\tau)+ = \gamma^\tau &\quad (\gamma^x)+ = -\gamma^x \\ (\gamma^y)+ = -\gamma^y &\quad (\gamma^z)+ = -\gamma^z \end{aligned} \tag{7}$$

where I is the unit matrix.

From these matrices, we deduce

$$\gamma_5 = \gamma^\tau\gamma^x\gamma^y\gamma^z \qquad (\text{with } (\gamma_5)^2 = -1) \tag{8}$$

and we set

$$\begin{aligned} \sigma^{jk} &= (i/2)(\gamma^j\gamma^k - \gamma^k\gamma^j) \\ {}'\sigma^{jk} &= \gamma_5\sigma^{jk} = (1/2)\varepsilon^{jkln}\sigma_{ln} \end{aligned} \tag{9}$$

where ε^{jkln} is completely antisymmetric and $\varepsilon^{\tau xyz} = 1$. Let us now introduce a 4-component wave vector $\Phi(\mathbf{x})$ and its associated $\overline{\Phi}(\mathbf{x})$=$\Phi^*(\mathbf{x})\beta$, where β is a 4×4 matrix defined by

$$(\gamma^j)^+\beta = \beta\gamma^j \qquad (\beta = \gamma^\tau). \tag{10}$$

With the help of these wave vectors, we define real local observables by mean values of the form $(\overline{\Phi}(\mathbf{x})O\Phi(\mathbf{x}))$ where O is one of the 16 operators belonging to the algebra. More precisely, we set

$$\begin{aligned} P &= (\overline{\Phi}\Phi) \\ J^j &= (\overline{\Phi}\gamma^j\Phi) \\ S^{jk} &= (\overline{\Phi}\sigma^{jk}\Phi) \\ K^j &= (\overline{\Phi}\ (-i\gamma_5\gamma^j)\Phi) \\ Q &= (\overline{\Phi}\gamma_5\Phi). \end{aligned} \tag{11}$$

We shall also use the notation

$$'S^{jk} = (\overline{\Phi}\ '\sigma^{jk}\Phi). \tag{12}$$

One can easily prove the following relations

$$\begin{aligned} J^j J_j &= -K^j K_j = P^2 + Q^2 \\ K^j J_j &= 0 \end{aligned} \tag{13}$$

and

$$\begin{aligned} S^{jn}S_{jn} &= 2(P^2 - Q^2) \\ J_j S^{jn} &= -QK^n \\ K_j S^{jn} &= -QJ^n \\ S_{jn}\ 'S^{jn} &= 4\ PQ \\ J_j\ 'S^{jn} &= PK^n \\ K_j\ 'S^{jn} &= PJ^n. \end{aligned} \tag{14}$$

P and Q are invariant, J^j is a component of a timelike vector $\mathbf{J}$, and K^j a component of a spacelike vector $\mathbf{K}$. Moreover, S^{jn} and $'S^{jn}$ are antisymmetric tensors. Using (13), we see that we can associate a timelike unit vector $\mathbf{T}$ with $\mathbf{J}$ and a spacelike unit vector $\mathbf{Z}$ with $\mathbf{K}$, by setting

$$\begin{aligned} J^j &= (P^2 + Q^2)^{1/2} T^j \\ K^j &= (P^2 + Q^2)^{1/2} Z^j. \end{aligned} \tag{15}$$

These vectors are orthogonal; we have

$$\mathbf{T}\cdot\mathbf{T} = 1 \qquad \mathbf{K}\cdot\mathbf{K} = -1 \qquad \mathbf{T}\cdot\mathbf{K} = 0. \tag{16}$$

We can now complete the system by introducing two spacelike unit vectors, $\mathbf{X}$ and $\mathbf{Y}$, which are orthogonal to each other and orthogonal to $\mathbf{T}$ and $\mathbf{Z}$.

$$\begin{aligned} &\mathbf{X}\cdot\mathbf{X} = -1 \qquad \mathbf{Y}\cdot\mathbf{Y} = -1 \qquad \mathbf{X}\cdot\mathbf{Y} = 0 \\ &\mathbf{X}\cdot\mathbf{T} = 0 \qquad \mathbf{X}\cdot\mathbf{Z} = 0 \qquad \mathbf{Y}\cdot\mathbf{T} = 0 \qquad \mathbf{Y}\cdot\mathbf{Z} = 0. \end{aligned} \tag{17}$$

For the time being, vectors $\mathbf{X}$ and $\mathbf{Y}$ are not completely defined, (only the plane $(\mathbf{X},\mathbf{Y})$ is defined), and therefore we may choose also for $\mathbf{X}$ and $\mathbf{Y}$, vectors ${}^T\mathbf{X}$ and ${}^T\mathbf{Y}$ with

$$\begin{aligned} {}^T\mathbf{X} &= \mathbf{X}\cos\delta - \mathbf{Y}\sin\delta \\ {}^T\mathbf{Y} &= \mathbf{X}\sin\delta + \mathbf{Y}\cos\delta \end{aligned} \tag{18}$$

where δ is arbitrary. This undefined δ is not an observable and will be interpreted later on, as corresponding to the phase of the wave function.

Now, equations (14) can be expressed in the following form (which can also be obtained in a specific representation)

$$S^{jk} = P(X^jY^k - Y^jX^k) - Q(T^jZ^k - Z^jT^k)$$
$$'S^{jk} = Q(X^jY^k - Y^jX^k) + P(T^jZ^k - Z^jT^k). \qquad (19)$$

Thus, we have eight parameters, two for P and Q and six for **T, Z, X** and **Y**, including the unphysical δ of equation (18). These degrees of freedom correspond to the eight parameters of Φ (four complex parameters) including the unphysical phase of Φ.

4 Dynamical equations and observables

The relativistic motion of an electron of mass m and charge e, in an electromagnetic field, can be described by means of the Dirac equation

$$\gamma^j(\hbar\,\partial_j + i\,ec^{-1}A_j)\Phi + imc\Phi = 0 \qquad (20)$$

where **A** is the potential vector. Its conjugate writes

$$(\hbar\,\partial_j\overline{\Phi} - i\,ec^{-1}A_j\overline{\Phi})\gamma^j - imc\overline{\Phi} = 0. \qquad (21)$$

Here and everywhere in the following, we write Φ and $\overline{\Phi}$ for $\Phi(\mathbf{x})$ and $\overline{\Phi}(\mathbf{x})$ (the same coordinates $\mathbf{x}$ appear in both cases).

First, it is easy to find a conservation equation. Let us multiply equation (20) by $\overline{\Phi}$ to the left and equation (21) by Φ to the right. By adding the resulting equalities, we obtain

$$\partial_j(\overline{\Phi}\gamma^j\Phi) = 0$$

which writes

$$\partial_j J^j = 0. \qquad (22)$$

This conservation law appears as a typical statistical mechanics equation [5].

We can now calculate **J**. Let us multiply (20) by $-i\overline{\Phi}\,\gamma_k$ to the left and (21) by $i\gamma_k\Phi$ to the right. By adding these equalities, we obtain

$$mc(\overline{\Phi}\gamma_k\Phi) + (\hbar/2)\partial^j(\overline{\Phi}\sigma_{jk}\Phi) - ec^{-1}A_k(\overline{\Phi}\Phi)$$
$$- \hbar(i/2)[(\overline{\Phi}\partial_k\Phi) - (\partial_k\overline{\Phi}\Phi)] = 0$$

which we write

$$mcJ_k = -(\hbar/2)\partial^jS_{jk} - ec^{-1}A_kP - \hbar C_k \qquad (23)$$

with

$$C_k = -(i/2)[(\overline{\Phi}\partial_k\Phi) - (\partial_k\overline{\Phi}\Phi)] \qquad (24)$$

where **C** is not an observable and will be transformed later.

The preceding calculation gives us four equations (23); we need four other ones, since we have eight parameters. Let us multiply equation (20) by $-\overline{\Phi}\gamma_5\gamma_k$ to the left and equation (21) by $\gamma_k\gamma_5\Phi$ to the right. By adding these equalities, we obtain

$$(\hbar/2)\partial^j(\overline{\Phi}(-i\gamma_5\sigma_{jk})\Phi) + ec^{-1}A_k(\overline{\Phi}\gamma_5\Phi)$$
$$- \hbar(i/2)[(\overline{\Phi}\gamma_5\partial_k\Phi) - (\partial_k\overline{\Phi}\gamma_5\Phi) = 0$$

which we write

$$-(\hbar/2)\partial^j\,{}'S_{jk} - ec^{-1}A_kQ - \hbar D_k = 0 \qquad (25)$$

with

$$D_k = -(i/2)[(\overline{\Phi}\gamma_5\partial_k\Phi) - (\partial_k\overline{\Phi}\gamma_5\Phi)] \qquad (26)$$

where **D** is not an observable and will be transformed later.

The constructed equations (23) and (25) (and also (22)) are not unique, because by multiplying (20) by $\overline{\Phi}O$ and (21) by $O\Phi$ where O is any observable, we could find many more equations, but they can be considered as sufficient and possessing the dynamical content of Dirac equation. These new equations are written in terms of observables and of quadrivectors **C** and **D** which will now be expressed in terms of P, Q, **T, X, Y** and **Z**.

5 Transformation of the quadrivectors C and D

C_k and D_k are not observables. However they can be expressed in terms of partial derivatives of observables, and therefore, we must calculate $\partial_k\Phi$. For this purpose, let us examine the connection which exists between a transformation of the wave function Φ ($\Phi(x) \to \Phi(x+\delta x)$), and the corresponding transformation of P, Q, **T, X, Y** and **Z**.

Firstly, an infinitesimal Lorentz transformation of the four vectors **T, X, Y** and **Z** is defined by the six component antisymmetric infinitesimal tensor δa^{jk}. Thus an arbitrary vector **B** transforms as follows

$$\delta B^j = \delta a^{jn}B_n. \qquad (27)$$

The vector **B** can be equal to **T, X, Y** or **Z**. Thus, as can be easily shown, transformation (27) is induced by the corresponding transformation of Φ and $\overline{\Phi}$;

$$\delta\Phi = -(i/4)\sigma_{jn}\Phi\;\delta a^{jn}$$
$$\delta\overline{\Phi} = (i/4)\overline{\Phi}\;\sigma_{jn}\;\delta a^{jn}. \qquad (28)$$

In this transformation, P and Q remain invariant.

However, the transformation of Φ may also lead to a modification of the invariants P and Q. An infinitesimal Lorentz transformation depends on six parameters whereas $\delta\Phi$ depends on eight parameters. Therefore we introduce the infinitesimal additional transformation

$$\delta\Phi = \Phi\;\delta b + \gamma_5\Phi\;\delta c$$
$$\delta\overline{\Phi} = \overline{\Phi}\;\delta b + \overline{\Phi}\gamma_5\;\delta c \qquad (29)$$

which leaves **T, X, Y** and **Z** invariant but which modifies P and Q, as shown below. We have

$$\begin{aligned}\delta P &= 2(P\ \delta b + Q\ \delta c)\\ \delta Q &= 2(Q\ \delta b - P\ \delta c)\end{aligned} \tag{30}$$

and also

$$\begin{aligned}\delta J^j &= 2J^j\ \delta b\\ \delta K^j &= 2K^j\ \delta b.\end{aligned} \tag{31}$$

The term $\delta\Phi = \Phi\ \delta b$ corresponds to a simple swelling; thus **X, Y, Z** and **T** remain invariant. The term $\delta\Phi = \gamma_5\Phi\ \delta c$ is more complicated: it changes the ratio of P and Q leaving P^2+Q^2 invariant; according to (31), **J** and **K** (*i.e.* **T** and **Z**) remain invariant in the transformation, and also the plane (**X, Y**) which is orthogonal to these vectors.
By combining equations (28, 29), we get

$$\begin{aligned}\delta\Phi &= -(i/4)\sigma_{jn}\Phi\ \delta a^{jn} + \Phi\ \delta b + \gamma_5\Phi\delta c\\ \delta\overline{\Phi} &= (i/4)\overline{\Phi}\sigma_{jn}\ \delta a^{jn} + \overline{\Phi}\ \delta b + \gamma_5\delta c\end{aligned} \tag{32}$$

and by replacing δ by ∂_k

$$\begin{aligned}\partial_k\Phi &= -(i/4)\sigma_{jn}\Phi\ \partial_k a^{jn} + \Phi\ \partial_k b + \gamma_5\Phi\ \partial_k c\\ \partial_k\overline{\Phi} &= (i/4)\overline{\Phi}\sigma_{jn}\ \partial_k a^{jn} + \overline{\Phi}\ \partial_k b + \overline{\Phi}\gamma_5\ \partial_k c.\end{aligned} \tag{33}$$

Putting these results in equations (24, 26), we obtain

$$\begin{aligned}C_k &= -(1/4)(\Phi\sigma_{jn}\Phi)\ \partial_k a^{jn} = (1/4)S_{jn}\partial_k a^{jn}\\ D_k &= -(1/4)(\overline{\Phi}\gamma_5\sigma_{jn}\Phi)\ \partial_k a^{jn} = (1/4){}'S_{jn}\ \partial_k a^{jn}.\end{aligned} \tag{34}$$

Let us now transform the preceding equations by replacing S_{jn} and ${}'S_{jn}$ by the expressions given in equations (19)

$$\begin{aligned}C_k &= -(1/4)[P(X_jY_n - Y_jX_n) - Q(T_jZ_n - Z_jT_n)]\ \partial_k a^{jn}\\ D_k &= -(1/4)[Q(X_jY_n - Y_jX_n) + P(T_jZ_n - Z_jT_n)]\ \partial_k a^{jn}.\end{aligned} \tag{35}$$

Let us now replace δ by ∂_k in equation (27)

$$\partial_k B^j = \partial_k a^{jn} B_n. \tag{36}$$

By using this expression for **X, Y, Z**, and **T** in equation (35), we obtain

$$\begin{aligned}C_k &= -(1/4)[P(X_j\partial_kY^j - Y_j\partial_kX^j)\\ &\quad - Q(T_j\partial_kZ^j - Z_j\partial_kT^j)]\\ D_k &= -(1/4)[Q(X_j\partial_kY^j - Y_j\partial_kX^j)\\ &\quad + P(T_j\partial_kZ^j - Z_j\partial_kT^j)].\end{aligned} \tag{37}$$

Then, we remark that, according to equations (17), we have

$$\begin{aligned}X_j\partial_kY^j + Y_j\partial_kX^j &= 0\\ T_j\partial_kZ^j + Z_j\partial_kT^j &= 0\end{aligned} \tag{38}$$

and we get finally

$$\begin{aligned}C_k &= -(1/2)[PX_j\partial_kY^j - QT_j\partial_kZ^j]\\ D_k &= -(1/2)[QX_j\partial_kY^j + PT_j\partial_kZ^j].\end{aligned} \tag{39}$$

These quantities are not observables but nevertheless they are expressed in terms of **T, Z, X** and **Y**, and this is exactly what we wanted.

6 Summary

Thus we transformed Dirac equation into relations involving only P, Q, **T, X, Y** and **Z**, (that is to say involving 8 functions).

We have the conservation equation (22)

$$\partial_j J^j = 0$$

where **J** is given by equation (23)

$$mcJ_k = -(\hbar/2)\partial_j S^{jk} - ec^{-1}A_kP - \hbar C_k$$

and by equation (25)

$$-(\hbar/2)\partial^j\ {}'S_{jk} - ec^{-1}A_kQ - \hbar D_k = 0$$

where J_k, S^{jk}, ${}'S_{jk}$, C_k and D_k are expressed as follows in terms of P, Q, **T, X, Y** and **Z** (Eqs. (15, 19, 37))

$$J^j = (P^2+Q^2)^{1/2}T^j \tag{40}$$

$$\begin{aligned}S^{jk} &= P(X^jY^k - Y^jX^k) - Q(T^jZ^k - Z^jT^k)\\ {}'S^{jk} &= Q(X^jY^k - Y^jX^k) + P(T^jZ^k - Z^jT^k)\\ C_k &= -(1/2)[PX_j\partial_kY^j - QT_j\partial_kZ^j]\\ D_k &= -(1/2)[QX_j\partial_kY^j + PT_j\partial_kZ^j]\end{aligned} \tag{41}$$

These equations are invariant for gauge transformations of the form (see Eq. (17))

$$A_k \to A_k - (\hbar c/e)\partial_k\alpha(\mathbf{x}) \tag{42}$$

$$\begin{aligned}X_j &\to X_j\cos[2\alpha(\mathbf{x})] - Y_j\sin[2\alpha(\mathbf{x})]\\ Y_j &\to X_j\sin[2\alpha(\mathbf{x})] + Y_j\cos[2\alpha(\mathbf{x})]\end{aligned} \tag{43}$$

which entails (since $X_jX^j = -1$)

$$\begin{aligned}C_k &\to C_k + P\ \partial_k\alpha(\mathbf{x})\\ D_k &\to D_k + Q\ \partial_k\alpha(\mathbf{x}).\end{aligned} \tag{44}$$

Remarks

1. Here we consider e as the elementary charge ($e < 0$ for an electron, $e > 0$ for a positron). Therefore it does not hurt our common sense to find e explicitly in an expression (Eqs. (23, 25)) which involves an external field (the potential vector **A**). Thus, there is no difficulty in defining $\alpha(\mathbf{x})$ (a number), as we did.
2. For a Dirac wave function, we note that we have always $J^\tau > 0$.

7 Electron in a time independent vector potential: energy level

When the potential vector does not depend on time and vanishes at infinity, the time dependence of the various symbols with respect to the energy can be easily obtained.

We see immediately from definition (11) that the observables are time and energy independent. On the other hand, from definitions (24, 26), we see that C_k and D_k are also time independent and that the contribution of the energy can be separated by writing (see Eqs. (24, 26))

$$\begin{aligned} C_j &= {}^\circ C_j - \delta_{j\tau} EP \\ D_j &= {}^\circ D_j - \delta_{j\tau} EQ \end{aligned} \tag{45}$$

which correspond to

$$\begin{aligned} X_j &= {}^\circ X_j \cos(-2Et) - {}^\circ Y_j \sin(-2Et) \\ Y_j &= {}^\circ X_j \sin(-2Et) + {}^\circ Y_j \cos(-2Et). \end{aligned} \tag{46}$$

The energy is an eigenvalue which is determined by the solutions of the equations.

8 Application: plane waves

For plane waves, all observables are constants. Therefore T_j = constant and the plane $(X,\ Y)$ remains fixed. Accordingly Z_j = constant. The equations reduce to

$$\begin{aligned} mcJ_k &= -\hbar C_k \\ D_k &= 0 \\ J_k J^k &= P^2 + Q^2 \qquad \text{(from Eq. (40)).} \end{aligned} \tag{47}$$

On the other hand

$$\begin{aligned} C_k &= -(P/2) X_j \partial_k Y^j \\ D_k &= -(Q/2) X_j \partial_k Y^j. \end{aligned} \tag{48}$$

The second equation (47) and the second equation (48) lead to

$$Q = 0$$

and consequently the third equation (47) reduce to

$$J_k J^k = P^2. \tag{49}$$

Let us set

$$\begin{aligned} X_n &= {}^\circ X_n \cos(-2k_j x^j) - {}^\circ Y_n \sin(-2k_j x^j) \\ Y_n &= {}^\circ X_n \sin(-2k_j x^j) + {}^\circ Y_n \cos(-2k_j x^j) \end{aligned} \tag{50}$$

where ${}^\circ X_n$ and ${}^\circ Y_n$ are constant. From these assumptions and from the first equation (47), we deduce

$$C_j = -P\ k_j$$

and by combining this result with the first equation (47)

$$mcJ_j = \hbar k_j P = p_j P$$

where p_j is a component of the momentum and where $p_\tau = E/c$. By putting this value of J_k in equation (49), we obtain (with $p^2 = p_x^2 + p_y^2 + p_z^2$)

$$(E^2/c^2) - p^2 = m^2c^2 \qquad \text{or } E = mc^2(1 + p^2/m^2c^2)^{1/2}.$$

Thus for a given value of p, we have two opposite values of E, a positive and a negative one. By rotation of the (X, Y) plane, we also obtain two opposite values of the spin.

9 Conclusion

In the preceding sections, the four complex components of Φ have been eliminated in favour of eight real local observables, and simple relations between these quantities have been found. From a conceptual point of view, the advantage is very great. Quantum mechanics seems to belong to the domain of statistical mechanics. From a practical point of view, the superposition principle seems to be lost (it is occulted), since observables are bilinear combinations of Φ and $\overline{\Phi}$. This difficulty may be considered as a serious drawback, but it should be realized that this superposition property is not essential (but very important), since it does not appear in classical mechanics.

From another point of view, the new equations may be more general than the Dirac equation. Indeed, such a situation generally occurs when, in equations, explicit expressions replace matrices, since these matrices correspond to special representations. Thus these new equations (which in some way look like Maxwell equations) may very well describe the motion of other particles and not only that of spin 1/2 particles.

Note that the wave function Φ and the matrices γ_j (of Dirac) have many different representations. On the contrary, equations obtained here are unique, and this is another advantage.

Thus, this article opens many questions which hopefully will be elucidated in further publications.

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5. We remark that a diffusion equation reads $\frac{\partial P}{\partial t} = \partial_k J$ with $J_k = \gamma(\partial_k P + \beta \partial_k VP)$ (where V is a potential and β the Boltzmann constant) which is somewhat equivalent to equation (22).